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XXX. *Geometrical Solutions of three celebrated Astronomical Problems, by the late Dr. Henry Pemberton, F. R. S. Communicated by Matthew Raper, Esq; F. R. S.*

LEMMA.

Read June 4, 1772. **T**O form a triangle with two given sides, that the rectangle under the sine of the angle contained by the two given sides, and the tangent of the angle opposite to the lesser of the given sides, shall be the greatest that can be.

Let [TAB. XII. Fig. 1.] the two given sides be equal to AB and AC: round the center A, with the interval AC, describe the circle CDE, and produce BA to E; take BF a mean proportional between BE and BC, and erect the perpendicular FG, and complete the triangle AGB.

Here the sine of BAG is to the radius, as FG to AG; and the tangent of ABG to the radius, as FG to FB: therefore, the rectangle under the sine of BAG and the tangent of ABG is to the square of the

the

Fig. 1.

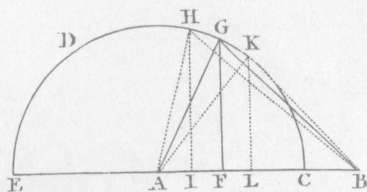


Fig. 2.

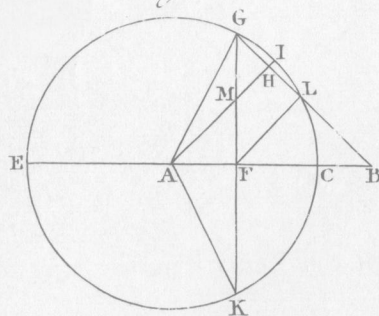


Fig. 4.

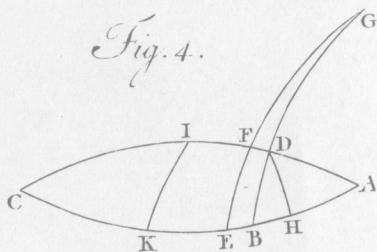


Fig. 5.

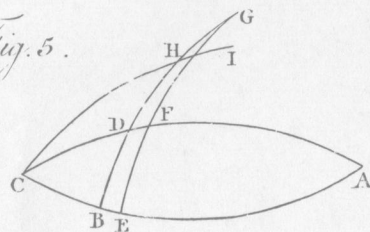


Fig. 7.

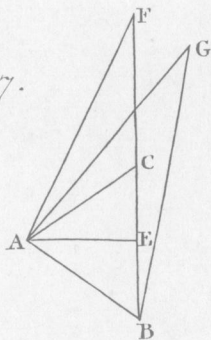


Fig. 8.

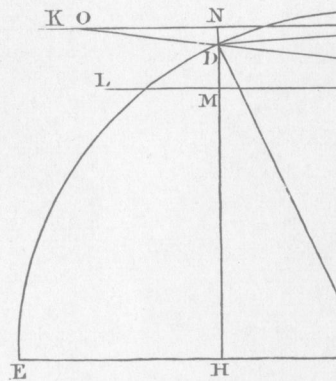
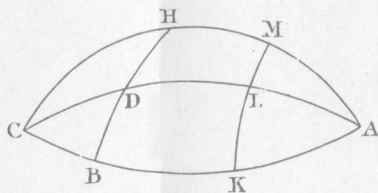


Fig. 2.

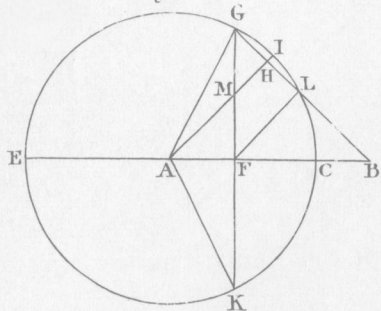


Fig. 3.

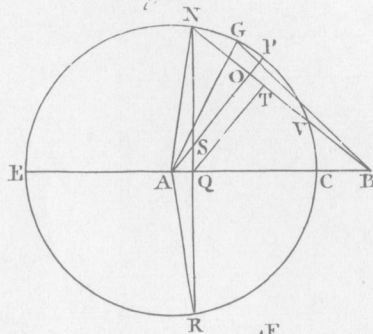


Fig. 5.

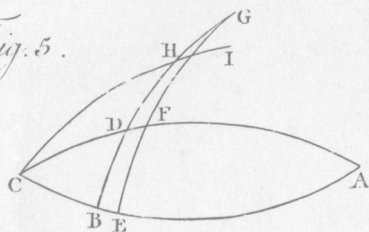


Fig. 6.

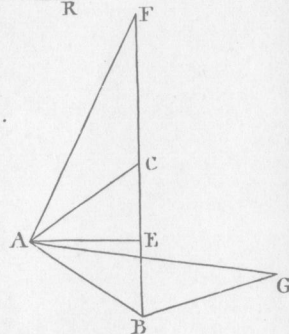


Fig. 8.

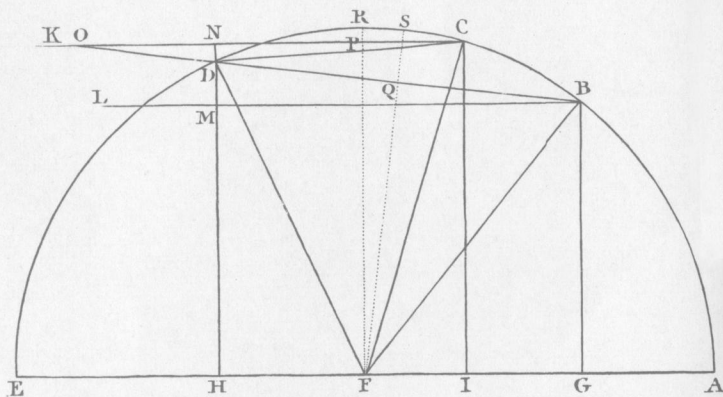
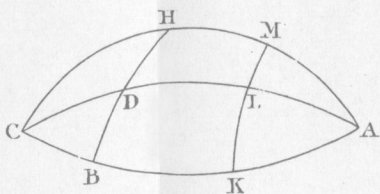


Fig. 9.

the radius, as the square of FG , or the rectangle EFC , to the rectangle under AG (or AC) and FB . But, EB being to BF as BF to BC , by conversion, EB is to EF as BF to FC , and also, by taking the difference of the antecedents and of the consequents, EF is to twice AF as BF to FC ; and twice AFB is equal to EFC .

Now, let the triangle BAH be formed, where the angle BAH is greater than BAG . Here, the perpendicular HI being drawn, the rectangle under the sine of BAH and the tangent of ABH will be to the square of the radius, as the rectangle EIC to the rectangle under AC , IB . But IF is to FB as $2AFI$ to $2AFB$, or EFC ; and $2AFI$ is greater than $AF^2 - AI^2$; also $AF^2 - AI^2$ together with EFC , is equal to EIC ; therefore, by composition, the ratio of IB to BF is greater than that of EIC to EFC ; and the ratio of $AC \times IB$ to $AC \times FB$ greater than that of EIC to EFC : also, by permutation, the ratio of $AC \times IB$ to EIC greater than the ratio of $AC \times FB$ to EFC . But the first of these ratios is the same with that of the square of the radius to the rectangle under the sine of BAH and the tangent of ABH ; and the latter is the same with that of the square of the radius to the rectangle under the sine of BAG and the tangent of ABG ; therefore, the latter of these two rectangles is greater than the other.

Again, let the triangle BAK be formed, with the angle BAK less than BAG , and the perpendicular KL be drawn. Then the rectangle under the sine of BAK and the tangent of ABK is to the square of the radius, as the square of KL to the rectangle under

$Kk k 2$

$AC,$

AC, BL. Here, FL being to FB as 2 AFL to 2 AFB or EFC, and 2 AFL less than $\text{AL}^2 - \text{AF}^2$, by conversion, the ratio of LB to FB will be greater than the ratio of ELC to EFC; therefore, as before, the rectangle under the sine of BAG and the tangent of ABG is greater than that under the sine of BAK and the tangent of ABK.

COROLLARY I.

BF is equal to the tangent of the circle from the point B; therefore, BF is the tangent, and AB the secant, to the radius AC, of the angle, whose cosine is to the radius as AC to AB. Therefore, AF is the tangent, to the same radius, of half the complement of that angle; and AF is also the cosine of the angle BAG to this radius.

COROL. 2.

The sine of the angle composed of the complement of AGB, and twice the complement of ABG, is equal to three times the sine of the complement of AGB. Let fall the perpendicular AH (Fig. 2.), cutting the circle in I; continue GF to K, and draw AK. Then $\text{BF}^2 = \text{EB} \cdot \text{C} = \text{GB} \cdot \text{L}$. Therefore, $\text{GB} : \text{BF} :: \text{BF} : \text{BL}$, and the triangles GBF, FBL are similar. Consequently FL is perpendicular to GB, and parallel to AH; whence GH being equal to HL, GM is equal to MF, and MK equal to three times GM.

Now, the arc $\text{IK} = 2 \text{ IC} + \text{GI}$; and the angle $\text{IAK} = 2 \text{ IAC} + \text{GAI}$; also GM is to MK as the

the sine of the arc GI to the sine of the arc IK, that is, as the sine of the angle GAI to the sine of the angle IAK. Therefore, the sine of the angle IAK ($= 2IAC + GAI$) is equal to three times the sine of the angle GAI; but GAI is the complement of AGB, and IAC the complement of ABG.

COROL. 3.

If (Fig. 3.) any line BN be drawn to divide the angle ABG, and AN be joined, also AO be drawn perpendicular to BN, and continued to the circle in P, the sine of the angle composed of NAP and $2PAC$ will be less than three times the sine of the angle NAP. Draw NQR perpendicular to AB, cutting AP in S; join AR, and draw QT perpendicular to BN, and parallel to AO; then $BQ^2 = NBT$. But BQ^2 is greater than the rectangle EBC, that is, greater than the rectangle NBV, under the two segments of the line BN drawn from B, to cut the circle in N and V: therefore, TB is greater than VB, and NO greater than OT. Consequently NS is greater than SQ. Hence RS is less than three times NS; and therefore, the sine of the angle PAR ($= NAP + 2PAC$) is less than three times the sine of NAP.

PROBLEM I.

To find in the ecliptic the point of longest ascension.

ANALYSIS.

Let (Fig. 4.) ABC be the equator, ADC the ecliptic, BD the situation of the horizon, when D is the point of longest ascension. Let EFG be another situation of the horizon. Then the ratio of the sine of EB to the sine of FD is compounded of the ratio of the sine of BG to the sine of GD , and of the ratio of the sine of AE to the sine of AF ; but the angles B and E being equal, the arcs EG , GB together make a semicircle; and, by the approach of EG towards GB , the ultimate magnitude of BG will be a quadrant, and the ultimate ratio of EB to FD will be compounded of the ratio of the radius to the sine of DG (that is, the cosine of BD) and of the ratio of the sine of AB to the sine of AD . Draw the arc DH perpendicular to AB . Then, in the triangle BDH , the radius is to the cosine of BD , as the tangent of the angle BDH to the cotangent of HBD . Also, in the triangle BDA , the sine of AB is to the sine of AD as the sine of the angle BDA (or BDC) to the sine of ABD ; therefore, the ultimate ratio of BE to DF is compounded of the ratio of the tangent of BDH to the cotangent of ABD , and of the ratio of the sine of BDC to the sine of ABD ; which two ratios compound that of the rectangle under the tangent of BDH and the sine of BDC to the rectangle under the cotangent and the sine of the given angle ABD .

But, when D is the point of longest ascension, the ratio of BE to DF is the greatest that can be; therefore, then the ratio of the rectangle under the tangent of BDH and the sine of BDC to the given rectangle under the cotangent and sine of the given angle ABD must be the greatest that can be; and consequently, the rectangle under the tangent of BDH, and the sine of BDC, must be the greatest that can be.

In the triangle BDA, the sine of BDH is to the sine of HDA, as the cosine of ABD to the cosine of BAD. Now, in the preceding lemma, let the angle BAG of the triangle AGB be equal to the spherical angle BDC: then will the sum of the angles ABG, AGB be equal to the spherical angle BDA. And, if AG in the triangle AGB, be to AB as the cosine of the spherical angle DBA to the cosine of DAB, that is, as the sine of BDH to the sine of HDA, the angle ABG, in the triangle, will be equal to the spherical angle BDH; and the angle AGB, in the triangle, equal to the spherical angle HDA. Therefore, by the first corollary of the lemma, that the rectangle under the tangent of the spherical angle BDH and the sine of BDC be the greatest that can be, the cosine of BDC must be equal to the tangent of half the complement of the angle, whose cosine is to the radius, as AG to AB, in the triangle, or as the cosine of the spherical angle ABD to the cosine of the spherical angle BAD.

If IK be the situation of the horizon, when the solstitial point is ascending, in the quadrantal triangle AIK, the cosine of KIC is to the radius as the cosine of IKA (\equiv DBA) to the cosine of IAK. Therefore,

fore, the cosine of BDC , when D is the point of longest ascension, is equal to the tangent of half the complement of the angle, which the ecliptic makes with the horizon, when the solstitial point is ascending.

But, the sine of the angle composed of DAB , and twice ABD , must be less than three times the sine of the angle BAD . In the spherical triangle ABD , the angles BAD , ABD together exceed the external angle BDC . Therefore, in the third corollary of the lemma, let the angle BAN be equal to the sum of the spherical angles BAD , ABD : but here, AN is to AB as the cosine of the spherical angle ABD to the cosine of BAD ; and AN is also to AB as the sine of ABN to the sine of ANB , that is, as the cosine of BAP to the cosine of NAP ; consequently, since the angle BAN is equal to the sum of the spherical angles BAD , ABD , the angle NAP is equal to the spherical angle BAD , and the angle BAP equal to the spherical angle ABD ; but the sine of the angle composed of NAP and twice PAB is less than three times the sine of NAP ; therefore, the sine of the angle composed of the spherical angle BAD and $2 ABD$ will be less than three times the sine of the angle BAD ; otherwise no such triangle DBA , as is here required, can take place, but the point A will be the point of longest ascension.

If the sine of the angle A be greater than one third of the radius, the point A can never be the point of longest ascension; but when the sine of this angle is less, the angle compounded of BAD and twice ABD , may be greater or less than a quadrant;
and

and therefore, the magnitude of the angle ABD, that A be the point of longest ascension, is confined within two limits, of which the double of one added to the angle A, as much exceeds a quadrant, as the double of the other added to that angle falls short of it; therefore, double the sum of those two angles, together with twice A, makes a semicircle; and the single sum of those two angles added to A makes a quadrant.

PROBLEM II.

To find when the arc of the ecliptic differs most from its oblique ascension.

ANALYSIS.

If (Fig. 5.) BD be the situation of the horizon, when CD differs most from CB, as before, the ultimate ratio of BE to DF will be compounded of the ratio of the radius to the sine of DG (or the cosine of DB) and of the ratio of the sine of CB to the sine of CD: but, when CD differs most from CB, BE and DF are ultimately equal; therefore, then the cosine of BD is to the radius as the sine of CB to the sine of CD.

Draw the arc CHI of a great circle, that DH be equal to DB; then, BH being double BD, half the sine of BH is to the sine of BD or DH, as the cosine of BD to the radius; therefore, half the sine of BH is to the sine of DH as the sine of CB to the sine of CD; but the sine of the angle BCH is to the sine of BH as the sine of the angle CHB to the

fine of CB ; whence, by equality, half the fine of BCH is to the fine of DH as the fine of CHB to the fine of CD : but as the fine of CHB to the fine of CD , so, in the triangle CHD , is the fine of DCH to the fine of HD : consequently, the fine of DCH is equal to half the fine of BCH . Hence, the difference of the angles BCH , DCH being given, those angles are given, and the arc CHI is given by position.

Moreover, in the triangle BCH , the base BH being bisected by the arc CD , the fine of the angle CHD is to the fine of the given angle CBD , as the fine of the given angle HCD to the fine of the given angle BCD ; therefore, the angle CHB is given; in so much, that in the triangle CBH all the angles are given.

The sum of the fines of the angles BCH , DCH is to the difference of their fines, as the tangent of half the sum of those angles to the tangent of half their difference; therefore, the tangent of half the sum of BCH , DCH is three times the tangent of half BCD .

In (Fig. 6.) the isosceles triangle ABC , let the angle BAC be equal to the spherical angle BCD , and let AE be perpendicular to BC ; also, CF being taken equal to CB , join AF : then EF is equal to three times EB ; and as EF to EB , so is the tangent of the angle EAF to the tangent of EAB ; but EAB is equal to half the spherical angle BCD : therefore, the angle EAF is equal to half the sum of the spherical angles BCD , BCH ; and consequently, the angle CAF equal to the spherical angle DCH . Here, AF is to CF as the fine of the angle ACF

to the sine of CAF ; and CB is to AB as the sine of the angle BAC to the sine of ACB : therefore, CF being equal to CB , and the sine of ACF to the sine of ACB , by equality, AF is to AB as the sine of the angle BAC to the sine of CAF , that is, as the sine of the spherical angle BCD to the sine of the spherical angle DCH .

Let (Fig. 7.) the triangle AGB have the angle ABG equal to the spherical angle CBD , and the side AG equal to AF . Then, AG is to AB as the sine of the spherical angle BCD to the sine of the spherical angle DCH , that is, as the sine of the spherical angle CBH to the sine of the spherical angle CHB : but AG is to AB also as the sine of the angle ABG to the sine of AGB ; therefore, the angle ABG being equal to the spherical angle CBH , the angle AGB is equal to the spherical angle CHB : and moreover, when the angle ABG is greater than ABF , that is, when the spherical angle CBH is greater than the complement of half BCD , the three angles ABG , AGB and BAC together exceed two right.

Hence, (Fig. 8.) towards the equinoctial point C , where the angle CBD is obtuse, a situation of the horizon, as BD , may always be found, wherein CD more exceeds CB than in any other situation: and when the acute angle DBA is greater than the complement of half BCD , another situation of the horizon, as KLM , may be found, toward the other equinoctial point A , wherein the arc of the ecliptic CK will be less than the arc of the equator, and their difference be greater than in any other situation. But, if the angle DBA be not greater than the complement

plement of half BCD , the arc of the ecliptic, between C and the horizon, will never be less than the arc of the equator, between the same point C and the horizon.

In the two situations of the horizon, the angles CHB and KMA are equal.

SCHOLIUM I.

To find the point in the ecliptic, where the arc of the ecliptic most exceeds the right ascension, is a known problem: that point is, where the cosine of the declination is a mean proportional between the radius and the cosine of the greatest declination.

In the preceding figure, supposing the angle CBD to be right, then, because when CD most exceeds CB , the cosine of BD is to the radius as the sine of CB to the sine of CD , and, in the triangle CBD , the sine of CB is to the sine of CD as the sine of the angle CDB to the radius, also the sine of CDB is to the radius as the cosine of BCD to the cosine of BD ; therefore, the cosine of BD is to the radius as the cosine of the angle BCD to the cosine of the same BD , and the cosine of BD is a mean proportional between the radius and the cosine of BCD .

SCHOLIUM 2.

In any given declination of the Sun, to find when the azimuth most exceeds the angle which measures the time from noon, is a problem analogous to the preceding.

Dr.

PROBLEM III.

*The tropic found, by Dr. Halley's method *, without any consideration of the parabola.*

The observations are supposed to give the proportions between the differences of the sines of three declinations of the Sun near the tropic ; but the sine of the Sun's place is in a given proportion to the sine of the declination ; therefore, the same observations give equally the proportion between the differences of the sines of the Sun's place, in each observation.

Now (Fig. 9.), let ACE be the ecliptic, AE its diameter between Υ and ϖ , and its center F ; let B, C, D be three places of the Sun ; BG, CI, DH the sines of those places respectively. Draw CK, BL parallel to AE, which may meet HD, in N and M. Then, by the observations, the ratio of DM to DN is given. Therefore, if BD be drawn to meet KL in O, the ratio of BD to OD is given ; and the ratio of BD to DC is also given, they being the chords of the given angles BFD, CFD : hence the ratio of CD to DO, in the triangle CDO, is given ; and consequently, the angle COD will be given : which angle is the distance of the tropic from the middle point of the ecliptic between B and D : for, FPR being perpendicular to OC, and FQS perpendicular to DB, the angle QFP is equal to QOP, the points O, P, Q, F, being in a circle.

* Vide Philosophical Transactions, N^o 215.

THE CALCULATION.

$$\left. \begin{array}{l} \text{DN} : \text{DM} \\ \text{f. } \frac{1}{2} \text{BFD} : \text{f. } \frac{1}{2} \text{CFD} \end{array} \right\} :: \text{rad.} : \text{t. } \angle \chi$$

$$\text{rad.} : \text{t. } \angle \chi \approx 45^\circ :: \text{t. } \frac{1}{4} \text{BFC} : \text{t. } \frac{\text{COD} \approx \text{DCO}}{2}$$

If $\chi > 45^\circ$, $\angle \text{COD} > \text{DCO}$

And

if $\chi < 45^\circ$, $\angle \text{COD} < \text{DCO}$.

If the intervals between the observations are so small, that the lines differ not much from the arches, the arches BC, CD may be counted in time, and the calculation may be abbreviated thus :

$$\text{DM} : \text{DN} :: \text{arc. BD} : \text{Z (for DO)}$$

$$\text{DC} + \text{Z} : 2 \text{DC} :: \frac{1}{4} \text{BC} : \text{SR.}$$

Or,

$$\text{DM} \times \text{DC} + \text{DN} \times \text{BD} : \text{DM} \times \text{DC} :: \frac{1}{2} \text{BC} : \text{SR.}$$